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non-equilibrium thermodynamics

The photon entropy production Σ_γ is an extensive quantity and thus is fundamentally a volume integral of its local density σ_γ . Re-expression of Σ_γ as a surface integral is effected through the divergence theorem and the surface photon entropy flux \mathbf{H} . This step requires the photons to propagate freely from a well-defined surface. Below this surface, the photons diffuse through matter instead.

In this case, of a star with a sharp surface of temperature T_{eff} and no atmosphere, the paragraphs surrounding equations (3.10) and (A.10) incorrectly present the surface integral form of Σ_γ (“the total boundary entropy production, Σ_{bound} ”) as an *additional* contribution to Σ_γ . In fact the boundary expression is simply the *total* Σ_γ within the enclosed volume, *including* all entropy production due to luminosity production and transport. The volume integrals (3.9) and (A.10) are the photon entropy production *up to, but not including*, the surface; the surface integral or boundary term (3.10) is the photon entropy production including the surface, as well as the interior.

Thus, the contribution of the surface *alone* to Σ_γ is the *difference* of these two expressions: $\Sigma_{\text{surface}} = \Sigma_{\text{bound}} - \Sigma_{\text{volume}}$. The use of the volume integral (3.9) alone in the variational principle (3.11) for thermal structure continues to be justified by the boundary conditions that hold the radius R and T_{eff} fixed.

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VARIATIONAL PRINCIPLES FOR STELLAR STRUCTURE

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ABSTRACT

The four equations of stellar structure are reformulated as two alternate pairs of variational principles. Different thermodynamic representations lead to the same hydromechanical equations, but the thermal equations require, not the entropy, but the temperature as the thermal field variable. Our treatment emphasizes the hydrostatic energy and the entropy *production rate* of luminosity produced and transported. The conceptual and calculational advantages of integral over differential formulations of stellar structure are discussed along with the difficulties in describing stellar chemical evolution by variational principles.

Subject headings: Stellar structure and evolution; variational principles; non-equilibrium thermodynamics

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1. Differential Equations of Stellar Structure

1.1. Static Stellar Structure as a Non-Equilibrium Steady State

In thermostatic equilibrium (Reichl 1980; Callen 1985), general statistical properties radically simplify a macroscopic description. Non-equilibrium systems, however, generally require a microscopic kinetic theory. In special non-equilibrium systems, however, it is possible to leave the microscopic physics implicit in statistical averages, and to proceed with a nominally macroscopic dynamics. Stable stars evolve quasi-statically through well-defined spatial structures and temporal stages, so that a macroscopic non-equilibrium thermodynamics can be formulated. In this paper, we apply non-equilibrium thermodynamics to stellar structure, as distinct from stellar evolution.

Macroscopic non-equilibrium thermodynamics began with the work of Rayleigh (1877) and Onsager (1931a, 1931b) (see also Casimir 1945); a complete theory was proposed by Prigogine and his followers which applies to an important but restricted class of systems (Prigogine 1945; Davies 1962; De Groot and Mazur 1962). Later developments have both extended (Donnelly *et al.* 1966; Glansdorff and Prigogine 1971) and rivaled this classic work (Tisza 1966; Truesdell 1969; Gyarmati 1970; Lavenda 1978). For stars, we may take a generalized Prigogine approach, assuming some type of local statistical equilibrium holds and intensive thermodynamics parameters are defined at least locally in space and time. This assumption validates a macroscopic approach.

Non-equilibrium thermodynamics is typically described by conjugate pairs of variables: *differences* in intensive parameters, the *thermodynamic forces*, and macroscopic, extensive *thermodynamic fluxes*. Usually one relates forces to fluxes as cause to effect and assumes a linear or quasi-linear relation between the two. In this paper, we describe a quasi-static star of steady luminosity as an open system, receiving energy from nuclear sources which is ultimately radiated out of the stellar surface.

For the mechanical and thermal structure of this non-equilibrium steady state (NESS), we present two different pairs of variational principles (equations (3.5) and either (3.11) or (4.2)). While the differential formulation of stellar structure integrates local quantities from point to point, either integral formulation directly starts with global properties, including, but not limited to, total mass, luminosity, and radius. Iterative application of a mesh approximation to the *global* variational integrals is analogous to the Henyey or relaxation *differential* method (Henyey, Vardya, and Bodenheimer 1965; Kippenhahn and Weigert 1990). However, such a discretization approximation is not necessary, and continuous analytic approximations are also possible, in terms of variational parameters of global significance.

1.2. Formal Similarity Between Mechanical and Thermal Equations

The four first-order differential equations of quasi-static stellar structure occur in two pairs: hydromechanical and thermal (Kippenhahn and Weigert 1990; Hansen and Kawaler 1994). In the Euler representation and assuming spherical symmetry and conductive transport of luminosity,

$$\begin{aligned} -dP/dr &= Gm\rho/r^2 \quad , \quad dm/dr = 4\pi r^2 \rho \quad , \\ -kdT/dr &= l/4\pi r^2 \quad , \quad dl/dr = 4\pi r^2 \rho(\varepsilon - \varepsilon_\nu) \quad , \end{aligned} \quad (1.1)$$

where the first horizontal pair is the mechanical (density-pressure) and the second the thermal (luminosity-temperature) equations. In radiative transport, the thermal conductivity k can be replaced by an equivalent radiative diffusion expression $k \rightarrow 4acT^3/3\kappa\rho$, where κ is the opacity of matter, $ac/4$ the Stefan-Boltzmann constant, and the Boltzmann constant k_B is set to unity throughout. (Convective forms of luminosity transport are discussed below.) The variables are: r = distance from center; $m(r)$ = cumulative mass from center to r , $P(r) = P_m(r) + P_\gamma(r)$ = total pressure (matter and radiation), $\rho(r)$ = matter density; $l(r)$ = photon luminosity at radius r , $T(r)$ = common matter-photon temperature, $\varepsilon(r)$ = luminosity production per unit mass, $\varepsilon_\nu(r)$ = neutrino luminosity per unit mass, G = Newton's gravitational constant. The luminosity is constant outside regions of luminosity production, where $\varepsilon = 0$.

Except in Section 5, we neglect the slow nuclear chemical *evolution*, so that the quasi-static stellar *structure* is in a NESS. Dropping the time derivatives, if the thermal conductivity were constant or $\propto T^{-2}$, equations (1.1-2) would be self-adjoint and would admit a variational principle formulation. Because the conductivity or radiation transport coefficients generally do depend on dependent variables, the thermal equations are not self-adjoint. A self-consistent variational principle is nevertheless still possible (section 3).

Except for this technical difference, the mechanical and thermal equations of quasi-static stellar structure are now formally symmetric: momentum transport and luminosity transport equations, luminosity conservation laws, transport coefficients and density are analogous. In this section, we exploit this structural similarity between mechanical and thermal equations, while still stressing the physical differences between the mechanical NESS and the steady photon entropy generation.

In the static Euler representation, the seven dependent variables are: m , P , ρ , T , l , κ , and ε . There are four differential and three constitutive equations; seven equations in all. With four first-order differential equations, there are four boundary conditions which are usually taken as $P(R) = 0$ (defining the surface $r = R$), $m(0) = 0$ (no mass singularity at the center), $m(R) = M$ (total mass of the star), and $T(R) = T_{\text{eff}}$ (surface temperature and

thus total luminosity).

Because the included mass is conserved and accumulates monotonically with increasing radius, a point transformation to m as the independent variable is possible. In this Lagrange representation, the position r of each mass shell dm is now dependent, $r = r(m)$. The structure equations now become:

$$\begin{aligned} -dP/dm &= Gm/4\pi r^4 \quad , \quad dr/dm = 1/4\pi r^2 \rho \quad , \\ -kdT/dm &= l/16\pi^2 r^4 \quad , \quad dl/dm = \varepsilon - \varepsilon_\nu \quad . \end{aligned} \quad (1.2)$$

The boundary conditions in the Lagrange representation assume M as given. Thus, a useful set of boundary conditions is $r(0) = 0$, $P(M) = 0$, $T(0) = T_c$, $P(0) = P_c$.

1.3. Differences Between Mechanical and Thermal Steady States

The first and third equations in the Lagrange representation show one difference between matter and radiation and between mechanical and thermal structure. The star is defined by the presence of opaque matter, but, unlike m , luminosity l is still a dependent variable. Because it does not increase outside the core, l would be useless as an independent variable in the outside regions which transport, but do not produce, luminosity.

A physically more important difference between matter and radiation arises because the matter is static so that matter thermodynamics enters only implicitly, through the equation of state $P_m = P_m(\rho, T)$, where the matter internal energy acts as a potential for the pressure. The thermal structure is not static, but shows a stationary flow of released nuclear energy from the core to the surface. Luminosity transport is explicitly statistical and depends explicitly on thermodynamic averages: the local temperature T and the opacity $\kappa(\rho, T)$. This non-equilibrium luminosity transport derives from non-equilibrium statistical mechanics (Appendix A). In radiative transport, this steady flow expresses the balance between the outward force of radiation and the resisting force of opacity (radiation friction):

$$-\frac{dP_\gamma}{dr} = \frac{\kappa\rho}{c} \frac{l}{4\pi r^2} \quad . \quad (1.3)$$

Convection is more complex than radiative transport and usually described in mixing length theory by the mixing length λ_{mix} as the transport parameter. The heat flux becomes:

$$l/4\pi r^2 = (\rho c_P T) \lambda_{\text{mix}}^2 \sqrt{g\gamma_P} \left[\frac{\nabla - \nabla_{\text{ad}}}{\lambda_P} \right]^{3/2} \quad , \quad (1.4)$$

for efficient convection, meaning that a convective cell loses little heat before it breaks up (Glansdorff and Prigogine 1971; Donnelly *et al.* 1966; Hansen and Kawaler 1994). (See

Appendix B for definitions.) Convection is efficient in two cases: The first is “slow but hot”, where ∇ is only slightly larger than ∇_{ad} (quasi-adiabatic regime), and the heat transport is efficient because the heat capacity c_P is large, but the cell velocity is slow. This regime occurs in the convective cores of stars with $M > 1.08M_{\odot}$. The second case is “warm but fast”, where ∇ is well above ∇_{ad} , c_P is not large, but the cell velocity is high. The outer convective layer of the Sun is in such a regime.

Convection only comes into play as a luminosity transport mechanism if the Schwarzschild instability criterion holds. Convection, radiative transport, and conduction can operate simultaneously and in the same region of the star; in which case, the total luminosity is the sum of all three transport fluxes. In the case of efficient convective heat transport, virtually the entire luminosity flux is convective.

The non-equilibrium steady-state obtains in quasi-static stages of stellar evolution, beginning with hydrogen and helium burning. Steady states, after the Main Sequence, burn carbon, neon, oxygen, or silicon, and are dominated by neutrino production and transport. The neutrinos escape essentially without matter interaction and without thermalizing. Neutrinos are produced by hot matter, but their energy loss is not thermostatic: their luminosity contributes to the material internal energy only by cooling the matter; they contribute nothing directly to stellar structure. The neutrino number and energy lost can only be inferred from the photon luminosity and the assumed matter properties and nuclear reactions (Bahcall 1989; Kippenhahn and Weigert 1990; Hansen and Kawaler 1994; Arnett 1996).

2. Stellar Structure as a Non-Equilibrium Steady State

No set of differential equations has a unique variational formulation (Douglas 1941). In this section and the next, we Legendre transform the global thermodynamic potentials, leaving the Euler-Lagrange equations invariant. The energy representation we start with enjoys a transparent physical interpretation in terms of hydrostatic energy and entropy production. In Section 4, we give entirely different variational principles in terms of local field variables, which nevertheless lead to the same mechanical and thermal equations (1.2).

2.1. Thermodynamic Equilibria

Thermodynamics distinguishes various types of equilibria or steady states (Gyarmati 1970; Glansdorff and Prigogine 1971; Reichl 1980; Callen 1985). *Global* mechanical or

thermal equilibrium implies uniform pressure and temperature everywhere, or at least throughout a large, finite system. Stars are not in global equilibrium; their pressures and temperatures vary in space, but are still meaningful locally in small regions. If the length scales over which kinetic mechanisms maintain local mechanical or thermal equilibrium (LME or LTE) are much smaller than the scale heights of pressure or temperature:

$$\lambda_P \equiv P/|\nabla P| \quad , \quad \lambda_T \equiv T/|\nabla T| \quad , \quad (2.1)$$

then a *local* pressure $P(r)$ or temperature $T(r)$ is well defined as a function of r . An equilibrium can be *partial* or *complete* among various constituents of the star. The LTE in a stellar interior is complete in that all matter species and radiation have a common temperature $T_m = T_\gamma = T$. The pressure equilibrium is partial, in that the total pressure is the sum of the partial pressures of each component, $P = P_m + P_\gamma$. The pressure decreases outwards because of the star's self-gravitation; the temperature decreases outwards wherever the total luminosity exceeds the neutrino luminosity.

The local material internal energy density is determined by the matter pressure P_m . In LTE, P_m is an equilibrium state function of temperature and matter density, $P_m = P_m(\rho, T)$, the matter equation of state. The photon equation of state is a function of T alone. In the luminosity equations, the transport coefficient (the opacity or conductivity) and the specific luminosity production, in LTE, are functions of local state variables T , ρ , and X_i , the local chemical composition (nuclear mass abundances X_i). The three constitutive functions depend on position only implicitly, through the local equilibrium variables $T(r)$, $\rho(r)$, $X_i(r)$. Since we are generally not considering chemical evolution, we generally suppress the dependence on X_i .

Both LME and LTE break down in the stellar atmosphere, where the matter becomes transparent and the radiation escapes almost unhindered. Here matter and radiation are not in thermal equilibrium. Pressure and gravity become unbalanced, as the star emits a stellar wind into space (Chandrasekhar 1950; Stix 1989)

To distinguish these local thermodynamic equilibria from global pressure and temperature equilibria, the global state is referred to as a *non-equilibrium steady state* (NESS). Along the Main Sequence and any later, post-Main-Sequence steady states, NESS obtains, with steady radial luminosity flow. Various time scales are associated with these local equilibria and NESS's, which must be achieved and maintained by different mechanisms. In the Sun, LME and LTE are achieved in about 10^{-12} sec by local kinetic mechanisms. The global mechanical and thermal NESS's are established in about 30 min and 10^7 yrs, respectively. These last two time scales characterize helioseismological disturbances and non-static, macroscopic heat flows (Kelvin-Helmholtz time). Except in dynamic stages, these time scales are much shorter than the local and global chemical

evolution times 10^{7-8} and 10^{10} yrs associated with nuclear transmutations (Bahcall 1989; Kippenhahn and Weigert 1990; Hansen and Kawaler 1994; Arnett 1996).

2.2. Different Thermodynamic Representations

Like any thermodynamic system, a star can be described by different global thermodynamic potentials. The original representation is the *energy representation* $E = E(V, S, N_i)$, as a function of volume V , entropy S , and species numbers N_i . Because the thermodynamic state varies spatially, the extensive state variables must be locally recast, as either spatial (per unit volume) or specific (per unit mass) densities (Chandrasekhar 1939; Kippenhahn and Weigert 1990). (We follow astrophysical custom by using the latter, unless otherwise noted.) Extensive quantities are then mass integrals over the specific densities. The specific energy e is the sum of the specific internal energy u and the specific gravitational energy $-Gm/r$. The specific volume v_ρ is $1/\rho$. In all but the hottest stars, the total entropy $S = S_m + S_\gamma$ is dominated by matter, with specific entropy s_m . Nevertheless, the small radiation entropy *flux* is responsible for the stellar luminosity and cannot be ignored.

In the stellar interior, where a NESS obtains, the specific entropy $s = s_m + s_\gamma$ is stationary, so that $E = E(V, S)$ and $e(\rho, s_m + s_\gamma)$ are, respectively, global and specific thermodynamic potentials. The radiation entropy density alone is determined by the local ratio $\kappa l/m$. If the thermal structure of the star is prescribed (e.g., isentropic or isothermal), a barotropic relation $P = P(\rho)$ holds so that the hydromechanical equations are closed. Nuclear burning in the stellar core leads to secular chemical changes (which we ignore in stellar structure), and to the rapid thermalization of fusion products, which steadily produces radiation entropy flux out of the star. In the steady state, the star's entropy is constant and cannot be used as a global thermodynamic state function. In order to treat matter and radiation as symmetrically as possible, we use the temperature T , which is common for the matter and radiation in the stellar interior. The appropriate generalization for the stellar atmosphere is straightforward. (See Appendix A, subsection 3.)

The thermodynamic potential for the new variables (V, T) is the Helmholtz free energy

$$F(V, T) = E(V, S) - TS \quad (2.2)$$

or specific Helmholtz free energy $f(\rho, T)$. (We might also go over to completely intensive state variables (P, T) and the Gibbs free energy $G(P, T) = F + PV$ as the thermodynamic potential. The choice of thermodynamic potentials is a matter of convenience; we prefer the Helmholtz free energy.)

3. Global Thermostatic Potentials and Entropy Production

In this section, we denote functional variations by δ , local thermodynamic state and spatial changes by ordinary differentials, so that the spatial differential $dF = \nabla F \cdot d\mathbf{r}$.

3.1. Mechanical Steady State

In considering only the mechanical NESS, we can use any thermodynamic representation. We can use the total hydrostatic energy (Lamb 1945; Chiu 1968; Rosenbluth *et al.* 1973; Hansen and Kawaler 1994):

$$E(V, S) = \int_0^M dm \, e(\rho(r), s(r)) \quad , \quad e = u - Gm/r \quad , \quad (3.1)$$

or the Helmholtz free energy

$$F(V, T) = \int_0^M dm \, f(\rho(r), T(r)) \quad , \quad f = e - Ts \quad , \quad (3.2)$$

as the thermodynamic potentials. The specific internal energy and free energy have variations

$$\delta e = -P\delta(1/\rho) + Gm\delta r/r^2 + T\delta s \quad , \quad (3.3)$$

$$\delta f = -P\delta(1/\rho) + Gm\delta r/r^2 - s\delta T \quad , \quad (3.4)$$

where $\delta(1/\rho) = d(4\pi r^2 \delta r)/dm$.

We could also use the global enthalpy

$$H(P, S) = E(V, S) + \int dV \, P \quad , \quad dH = VdP + TdS \quad ,$$

or the Gibbs free energy

$$G(P, T) = E(V, S) + \int dV \, P - \int dS \, T \quad , \quad dG = VdP + SdT \quad ,$$

as global potentials. The hydrostatic equation (1.2) is obtained by freezing the thermal structure, taking adiabatic variations in E or H or isothermal variations in F or G . For either mechanical variation, $\delta P/\rho = \delta h$ or δg . In all cases, the functional variation does not assume that the frozen thermal structure is in the NESS. But the short mechanical time scale compared to the thermal time scale assures that LTE, although not necessary, is a sufficient condition for LME. “Isothermal” here refers not to *spatially constant* temperature, but *unvaried* temperature *profile*; similarly, “adiabatic” refers to unvaried specific entropy profile.

The simplest example of the resulting Euler-Lagrange equations arises from applying the variation (3.3) to (3.1), setting $\delta s = 0$ (Chiu 1968). Using $\delta(1/\rho) = d(4\pi r^2 \delta r)/dm$ and integrating the first term in (3.3) by parts, one obtains

$$\delta E = \int_0^M dm \left[4\pi r^2 (dP/dm) + Gm/r^2 \right] \delta r = 0 \quad , \quad (3.5)$$

ignoring the boundary condition terms. Setting the integrand here to zero for arbitrary δr reproduces the hydrostatic NESS, the first equation of (1.2). Alternatively, applying the variation (3.4) to (3.2) isolates the mechanical structure in the isothermal limit $\delta T = 0$.

3.2. Entropy Production in the Thermal Steady State

In our NESS, the entropy density of matter and of radiation are each stationary and very disparate in magnitude. Because matter and radiation share a common temperature, we use T as the intensive thermodynamic variable. Instead of the entropy, our variational principle for thermal NESS minimizes the *entropy production rate* $\Sigma \geq 0$ (Prigogine 1945; Davies 1962; De Groot and Mazur 1962; Gyarmati 1970; Donnelly *et al.* 1966; Glansdorff and Prigogine 1971). A simple minimum entropy production variational principle applies to systems close to global equilibrium in the linear regime and to a special class of systems, those in the quasi-linear regime. In the linear regime, transport coefficients are constant. In the quasi-linear regime, LTE still obtains, but the constitutive functions depend only on the local state variables and not on their derivatives. Because the stellar NESS is in the nonlinear regime, the thermal variational principle depends on two temperatures, T , T_* , the first of which is dynamical, the second referring to the background matter.

The entropy production rate is bilinear, a product of thermodynamic forces and fluxes. For temperature variations, the thermodynamic force arises, not from T , but from differences of $1/T$, or from the gradient of $1/T$. In the linear or quasi-linear regime, the luminosity flux is linear in $\nabla(1/T)$. This thermal gradient drives the flow of luminosity from point to point, for any of the three standard heat transport mechanisms.

The temperature also occurs in other parts of the entropy production rate, but only as a *local* state variable having nothing to do with heat transport. We must distinguish this temperature, T_* , from the temperature T which is subject to functional variations (Donnelly *et al.* 1966; Glansdorff and Prigogine 1971). With this distinction, we can write the entropy production of conductive transport:

$$\Sigma_{\text{cond}} = \int dV \frac{1}{2} (kT^2)_* [\nabla(1/T)]^2 \quad , \quad (3.6)$$

and of radiative diffusion as:

$$\Sigma_{\gamma \text{ diff}} = \int dV \frac{1}{2} \left(\frac{4acT^5}{3\kappa\rho} \right)_* [\nabla(1/T)]^2 \quad , \quad (3.7)$$

with details in Appendix A (Chandrasekhar 1950; Essex 1984a, 1984b). The entropy production of convective transport is presented in Appendix B (B.2). Any transport entropy production is a measure of the efficiency of these different mechanisms for luminosity transport, the price paid for transporting the luminosity produced in the core of the star to the surface.

The entropy production due to nuclear burning in the core is:

$$\Sigma_{\text{fusion}} = [\varepsilon\rho]_*/T \quad . \quad (3.8)$$

The bulk radiation entropy production rate is the sum of the luminosity transport and production terms (3.6-8):

$$\begin{aligned} \Sigma_{\gamma} &= \Sigma_{\gamma \text{ diff}} + \Sigma_{\text{fusion}} \\ &= \int dV (1/2) [4acT^5/(3\kappa\rho)]_* [\nabla(1/T)]^2 + (\varepsilon\rho)_*/T \quad , \end{aligned} \quad (3.9)$$

in case of radiative transport.

Where the radiation outstreams from the photosphere, at temperature T_{eff} ,

$$\Sigma_{\text{bound}} = \frac{4}{3} (ac/4) T_{\text{eff}}^3 (4\pi R^2) \quad (3.10)$$

is obtained by integrating the radiation entropy flux over the surface of the star. Because the mechanical NESS is reached so much faster than the thermal NESS, the mechanical structure must be the hydrostatic NESS corresponding to a given thermal distribution, stellar radius R and surface temperature T_{eff} (Sieniutycz and Berry 1989, 1991, 1992, 1993; Sieniutycz and Salamon 1990; Sieniutycz 1994). This boundary entropy production is held fixed by the boundary conditions and plays no dynamical role in the thermal variational principle.

3.3. Minimum Entropy Production

Our thermal variational principle requires minimizing the entropy production by varying $T \rightarrow T + \delta T$, but not T_* , holding M and T_* fixed (Donnelly *et al.* 1966). Only after the Euler-Lagrange equation is obtained, is T_* set equal to T . For each of the transport

entropy productions (3.6, 3.7, B.2), the standard heat transport equations are obtained in this way. For example, the radiative diffusion case is derived from (3.9):

$$\delta\Sigma_\gamma = \int dV \left[\left[\frac{4acT^5}{3\kappa\rho} \right]_* [\nabla(1/T) \cdot \nabla\delta(1/T)] - [\varepsilon\rho]_* (\delta T/T^2) \right] = 0 \quad . \quad (3.11)$$

Integrating the first term by parts and dropping the boundary terms, we set T_* to T and the entire integrand to zero. For arbitrary variations δT , the result is radiative diffusion:

$$\frac{d}{dr} \left(\frac{16\pi r^2 acT^3}{3\kappa\rho} \cdot \frac{dT}{dr} \right) + 4\pi r^2 \varepsilon\rho = 0 \quad . \quad (3.12)$$

For simplicity, only the radial dependence has been kept, and the neutrino luminosity ε_ν has been ignored. This same result is obtained by combining the thermal-luminosity pair of equations in (1.1).

In practice, an ansatz is made for $T_*(r)$ and for $T(r)$, with only the latter containing variational parameters to be determined by minimizing the variational integral. The procedure is iterative: after each step $T_*(r)$ is set equal to the $T(r)$ obtained at the previous step. One may use a global mesh, analogous to that in the *differential* Henyey method, or an analytic form with a finite or infinite number of adjustable and physically suggestive parameters. The procedure is identical to the background field (BF) or self-consistent field (SCF) method used in quantum many-body and field systems and can lead to an analytic approximation for the stellar structure in terms of global parameters. These may be related to global properties and boundary conditions of the star.

Because no entropy production is associated with the thermalization or diffusion of neutrinos, the only neutrino entropy production is by averaging over neutrino energies produced by the thermalized reactant matter. The exact expression for neutrino entropy production varies by reaction, but for high temperatures, its dimensional order of magnitude is very approximately:

$$\Sigma_\nu \sim \dot{N}_\nu \quad ,$$

where \dot{N}_ν is the total neutrino production rate. (A better estimate is given in Appendix C, C.10) The neutrino contribution to entropy production is usually much smaller than the photonic. But in advanced stages of stellar evolution, the temperature is high enough to bring weak interactions significantly into play, letting neutrino entropy production rival that of heat and radiative diffusion mechanisms.

In supernova explosions, neutrinos do play the same role as photons in ordinary stars: they interact with the ambient stellar matter, are thermalized, and are then emitted from the surface of the neutrinosphere at a definite blackbody temperature (Bahcall 1989; Arnett

1996). The temperature in the innermost region reaches a very high 30-50 MeV, so that neutrino luminosity transport is efficient and neutrino entropy production remains fairly small, below that of other energy transport mechanisms.

4. Variational Principles with Local Specific Potentials

In the preceding section, we presented variational principles for the hydrostatic NESS (3.1-5) and for steady-state luminosity transport (3.6-8). For mechanical NESS alone, any global thermodynamic potential can be extremized, but for the thermal steady state, only the global potentials, $G(P, T)$ or $F(V, T)$, and a close relative, $\Sigma(T)$, could be used. Independently of thermodynamics, we know that the same differential equations admit of many different variational integral forms. Because the specific potentials, potentials per unit mass, vary in space and also represent the local thermodynamic state just as well as the variables (V, P, T, S) , we may take the specific potentials as field variables and find other variational integrals to serve as global actions.

In the mechanical equations, we can switch from the density ρ and pressure P as the intensive state variables to either the specific enthalpy

$$h(P, s) = e(\rho, s) + P\rho \quad , \quad dh = \rho dP + Tds \quad (4.1a)$$

or the specific Gibbs free energy

$$g(P, T) = e(\rho, s) + P\rho - Ts \quad , \quad dg = \rho dP + sdT \quad , \quad (4.1b)$$

as the new local field variables. The mechanical NESS is obtained by either varying

$$\mathcal{I}_{\text{ad}} = \int dV \left[-\left(\frac{1}{8\pi G}\right)(\nabla h)^2 + P(h) \right] \quad (4.2a)$$

adiabatically or

$$\mathcal{I}'_{\text{iso}} = \int dV \left[-\left(\frac{1}{8\pi G}\right)(\nabla g)^2 + P(g) \right] \quad , \quad (4.2b)$$

isothermally, so that $\delta P/\rho =$ either δh or δg .

For the thermal part, we define a *heat or radiation potential density* θ , where $d\theta = \mathbf{J}_Q \cdot d\mathbf{r}$ and $d\mathbf{r}$ is the distance increment through which the luminosity is transported. In conductive or radiative transport, $d\theta = -k dT$ or $-(c/\kappa\rho)dP_\gamma$. In convective transport,

$$d\theta = (\rho c_P T)(\lambda_{\text{mix}}/\lambda_P)[\nabla - \nabla_{\text{ad}}](w dr) \quad , \quad (4.3)$$

where w = upward speed of the convective cell, assuming $\nabla \geq \nabla_{\text{ad}}$ (see Appendix C). The luminosity source is included by a *source potential* $\Pi(\theta)$, where $d\Pi = \rho \varepsilon d\theta$. The thermal variational integral is then:

$$\mathcal{J} = \int dV \left[-\left(\frac{1}{2c}\right)(\nabla\theta)^2 - \Pi(\theta) \right] \quad , \quad (4.4)$$

where r is the independent variable and θ the field. Variation of (4.4), keeping the mechanical structure fixed, yields the thermal transport equations for all three kinds of transport. These actions \mathcal{I}, \mathcal{J} do not have the simple thermodynamic interpretation that the minimal entropy production principle enjoys.

5. Variational Thermohydrodynamics and Chemical Evolution

A full statement of stellar *evolution* using entropy production and variational principles is beyond the scope of this paper, and only a sketch is presented in this section. *Evolution* refers only to nuclear chemical evolution, not to faster hydrodynamic or thermal changes with fixed nuclear abundances (Kippenhahn and Weigert 1990; Arnett 1996). The latter two processes are only time-dependent versions of the hydrostatic and thermal NESS. Such dynamical but non-evolutionary behavior can be treated with time-dependent extensions of the static variational principles stated in section 3. The hydrodynamic part is derived by minimizing the action (Lamb 1945; Sieniutycz and Salamon 1990; Sieniutycz 1994):

$$\int dt \int dm \left[\frac{\partial\Phi}{\partial t} - \frac{1}{2}(\nabla\Phi)^2 + e(\rho, s) \right] \quad , \quad (5.1)$$

where Φ is the velocity potential, $\mathbf{v} = -\nabla\Phi$ = the macroscopic velocity of fluid flow. This action principle is adiabatic and holds only for irrotational fluid flow. The isothermal analogue of (5.1) can be obtained by substituting $f(\rho, T)$ for $e(\rho, s)$. (A hydromechanical small-oscillation variational principle has been derived for asteroseismology by Backus and Gilbert 1967). The non-steady thermal behavior can be derived by minimizing the variational integral (Donnelly *et al.* 1996; Glansdorff and Prigogine 1971):

$$\int dt \int dV \left[\sigma_{\text{cond}} + \sigma_{\gamma \text{ diff}} + \sigma_{\text{conv}} + \frac{\dot{q}_*}{T} + [\rho c_V T^2]_* \frac{\partial}{\partial t} \left(\frac{1}{T} \right) \right] \quad , \quad (5.2)$$

where c_V is the specific heat capacity at constant volume, assuming the mechanical part to be instantaneously in the NESS (Sieniutycz and Salamon 1990; Sieniutycz 1994). Equation (5.2) is not a minimum entropy production principle, because it represents a NESS only if $\partial T / \partial t = 0$.

The two variational principles (5.1,2) can be used to treat stellar oscillations, with or without linearization. In general, these principles encompass only the propagation of seismic waves, not their driving forces, unless the latter are explicitly included in the integrals.

5.1. Gravitational Settling in Chemical Steady State

Stellar evolution involves two fundamental changes: gravitational settling of heavier nuclei arising from spatially inhomogeneous nuclear fusion reactions and thermonuclear transmutation of elements. In the Sun, for example, the associated global time scales are 6×10^{13} and 10^{10} yrs, respectively. Both evolution processes can be described in terms of entropy production, but only element diffusion is close to local chemical equilibrium (Kippenhahn and Weigert 1990; Bahcall and Pinsonneault 1992). As the mechanical and thermal timescales are usually much shorter than the chemical timescale, the star is at each instant in evolution in both hydrostatic and thermal NESS. But the boundary conditions for these NESS's change with chemical evolution, as the star's mass, radius, and luminosity change.

Chemical diffusion by gravitational settling can be dynamically formulated in terms of a minimum entropy production $\Sigma_{\text{nuc diff}}$ (Donnelly *et al.* 1966; Glansdorff and Prigogine 1971):

$$\begin{aligned} \Sigma_{\text{nuc diff}} = \int dV \left[\sum_{ij} \frac{1}{2} [\mathcal{D}_{ij}(\rho, T)]_* \nabla \left(\frac{\mu_i - \mu_H}{T} \right) \nabla \left(\frac{\mu_j - \mu_H}{T} \right) \right. \\ \left. + \sum_i [\mathcal{D}_{iT}(\rho, T)]_* \nabla \left(\frac{\mu_i - \mu_H}{T} \right) \nabla \left(\frac{1}{T} \right) \right] , \end{aligned} \quad (5.3)$$

where the $\mu_i(\mathbf{r}, t)$ are chemical potentials of the nuclear species i , and the $\mathcal{D}_{ij}(\rho, T)$ are the chemical diffusion coefficients, including cross terms between different species. The mixed thermodiffusion effect is included with the \mathcal{D}_{iT} terms. By Onsager's theorem, $\mathcal{D}_{ij} = \mathcal{D}_{ji}$ (Onsager 1931a, 1931b). Each chemical potential is defined relative to some reference potential, here taken to be that of hydrogen (H). This form of chemical diffusion assumes a quasi-linear relation between the element flux \mathbf{J}_i and the chemical gradients $\nabla \mu_i$:

$$\mathbf{J}_i = - \sum_j \mathcal{D}_{ij}(\rho, T) \nabla \left(\frac{\mu_i - \mu_H}{T} \right) - \mathcal{D}_{iT}(\rho, T) \nabla \left(\frac{1}{T} \right) , \quad (5.4)$$

in analogy with heat conduction. The diffusion represented by (5.3) is therefore in a NESS, as equation (5.4) is time-independent.

Chemical diffusion could also be formulated in terms of the extensive variables N_i , the

species densities n_i , or the nuclear mass abundances X_i , by use of the *grand potential*:

$$\begin{aligned} A(V, T, \mu_i) &= F - \int dV \sum_i \mu_i n_i \quad , \\ n_i &= -\rho(\partial a / \partial \mu_i) \quad , \end{aligned} \quad (5.5)$$

where $a(\rho, T, \mu_i)$ is the specific grand potential.

5.2. Thermonuclear Burning as Nonlinear Reactions

The entropy production of nuclear fusion is composed of a radiation part Σ_γ , already discussed in section 3, and a matter part Σ_{matter} , hitherto ignored, because it is significant only in late stages of stellar evolution. Starting with the Gibbs form of the entropy increment (De Groot and Mazur 1962; Gyarmati 1970; Glansdorff and Prigogine 1971), the canonical expression for Σ of matter undergoing chemical/nuclear reactions is:

$$\Sigma_{\text{matter}} = \int dV \sum_i \frac{\dot{n}_i \mu_i}{T} \quad . \quad (5.6)$$

where n_i are the number densities of each species i . The photon chemical potential μ_γ is zero, because photon number is not conserved. The neutrino chemical potential μ_ν is zero as long as neutrinos have no significant density.

A more transparent form can be obtained by defining common reaction rate densities $\dot{\xi}_\alpha = \dot{n}_i / \nu_{\alpha,i}$, where the subscript α labels the reaction. The $\nu_{\alpha,i}$ are the stoichiometric coefficients of reaction α :

$$\nu_{\alpha,1}N_1 + \nu_{\alpha,2}N_2 + \dots \rightarrow -\nu_{\alpha,j+1}N_j - \nu_{\alpha,j+2}N_{j+2} + \dots \quad ,$$

for all participating reactants $i = 1 \dots j$ and products $i = j + 1 \dots$. The *affinity* \mathcal{A}_α of reaction α :

$$\mathcal{A}_\alpha = \sum_i \nu_{\alpha,i} \mu_i \quad (5.7)$$

is a measure of how far a given reaction α is from chemical equilibrium. For a dead star (one whose nuclear reactions have gone to completion), $\mathcal{A}_\alpha = 0$ for all α . The alternate form for Σ_{matter} is then:

$$\Sigma_{\text{matter}} = \int dV \sum_\alpha \frac{\mathcal{A}_\alpha \dot{\xi}_\alpha}{T} \quad , \quad (5.8)$$

a canonical bilinear form in the forces \mathcal{A}_α/T and fluxes $\dot{\xi}_\alpha$.

Because $\mathcal{A}_\alpha \sim \text{MeV}$ is higher than the ambient temperature T , the thermodynamic forces driving nuclear burning are large. This makes stellar thermonuclear reactions highly

nonlinear, even locally (Reichl 1980). Because no general linear or quasi-linear relation holds between the fluxes and the forces in the matter contribution to Σ , without additional knowledge or details, we can go no further in expressing the entropy production of matter due to nuclear fusion than the kinematical expressions (5.6, 8). Without a quadratic (or any other expression) for Σ_{matter} in terms of the forces \mathcal{A}_α/T there is no automatic principle of minimum entropy production (Prigogine 1945; Lavenda 1978).

A variational principle of the quadratic type requires a quasi-linear relation between forces and fluxes. A non-quadratic minimum entropy production principle for matter fusion may still be possible, if the chemical evolution is in a NESS, with the fluxes $\dot{\xi}_\alpha$ constant in time. This NESS approximation holds as long as the star remains on one branch of hydrostatic thermonuclear burning (H, He, C, Ne burning), or on one of the late, hydrodynamic stages of chemical evolution (O, Si burning) (Arnett 1996). Furthermore, each burning stage constitutes a different NESS. It is possible that minimum Σ_{matter} holds on each burning stage, but that the bifurcation from one burning stage to the next is a discontinuous phase transition similar to the switch from non-convective to convective heat transport (Appendix B). So long as some fuel remains locally from one stage, that burning continues; the minimum entropy production conjecture would imply that only when its fuel runs out is the next burning stage preferred on entropy production grounds. Similar considerations may also hold for the evolution of protostars.

If the matter fusion is not representable as a NESS, another, time-dependent variational principle, analogous to (5.2), may still be possible, lacking a simple interpretation as minimum entropy production.

6. Summary and Outlook

Variational principles are most useful in expressing general properties of static or dynamical systems, such as exact or approximate symmetries, and in suggesting generalized coordinates. Integral principles also open alternative paths to solution via an iteration of approximate solutions. Under some restrictions, such methods converge and provide a useful replacement for numerical differential methods (Donnelly *et al.* 1966).

We have presented two different pairs of variational principles to replace the standard four equations of a hydrostatic NESS and steady luminosity flow. Two of these, the thermal variational principles (3.6-8) and (4.4) for NESS luminosity transport, are the principal original results of this paper. These represent a global alternative to numerical integration of the four differential equations of stellar structure. Practically, these thermal variational

principles suggest calculational procedures analogous to either the relaxation technique or the Rayleigh-Ritz method (Donnelly *et al.* 1966; Glansdorff and Prigogine 1971). Examples of global analytic approximations will be published elsewhere.

As seen in section 5, the static variational integrals of sections 3 and 4 can be extended to the time-dependent hydrodynamic and thermal non-steady states.

We have been able to recast stellar evolution only partially into integral form, and a dynamical principle is lacking. A complete integral reformulation of stellar theory would encompass these outstanding issues: the transmutation of elements, the emission of neutrinos, and bifurcations and instabilities during the multiple burning stages.

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Appendix A: Conductive and Radiative Entropy Production

Let us define notation and simple concepts first. LTE is assumed to be valid in the strictly local limit, with small deviations over small distances. Thus matter and radiation are in the maximal entropy state locally, with non-zero second-order thermodynamic fluctuations about the LTE. Purely local state variables and functions carry an asterisk (*) subscript. These are not subject to the non-equilibrium functional variations (Donnelly *et al.* 1966; Glansdorff and Prigogine 1971; Chandrasekhar 1939 and 1950; Essex 1984a, 1984b; Holden and Essex 1996).

1. Heat Conduction

The familiar case of heat conduction best introduces the entropy production Σ . This function's spatial density σ takes on, for matter alone, a simple bilinear form derived from the Gibbs equilibrium formula for the entropy increment $dS = dQ/T + \text{mechanical, chemical, ... etc. terms}$ (Callen 1985; Balian 1992). The entropy produced by a heat current

\mathbf{J}_Q flowing through a temperature field T in a volume V with a surface area A is:

$$\begin{aligned}\Sigma_{\text{cond}} &= \int d\mathbf{A} \cdot (\mathbf{J}_Q/T) \\ &= \int dV \nabla \cdot (\mathbf{J}_Q/T) \quad .\end{aligned}\tag{A.1}$$

In a steady state, $\nabla \cdot \mathbf{J}_Q = \dot{q}$, where \dot{q} is the heat source density within the volume. On the other hand, \mathbf{J}_Q in the quasi-linear case is linear in an externally given inverse-temperature gradient, with:

$$\mathbf{J}_Q = (kT^2)\nabla(1/T) \quad ,\tag{A.2}$$

where $k(\rho, T)$ is the local thermal conductivity, an integral over the microscopic momentum space:

$$kT^2 = (4\pi/3) \int_0^\infty dp \, p^2 \, f_0(\mathbf{r}, \mathbf{p}) [v^2(\rho e)^2/w(p)] \quad ,\tag{A.3}$$

where v and p are the atomic speed and momentum, ρe the gas internal energy density, f_0 the reduced 1-particle phase space distribution at LTE, and $w(p)$ the collision time (Balian 1992). Thus:

$$\sigma_{\text{cond}} = (kT^2)_* [\nabla(1/T)]^2 + \dot{q}_*/T\tag{A.4}$$

is the entropy density production rate by the transport and production of heat.

In (A.4) the current \mathbf{J}_Q must be evaluated assuming a *given* external T gradient. In reality, this gradient is produced by the current itself. Thermodynamically, the form (A.4) assumes an externally-imposed *first-order* deviation in entropy from LTE, while in fact, the deviation for the transport term is a *second-order* fluctuation (Donnelly *et al.* 1966; Glansdorff and Prigogine 1971). (LTE makes the first-order fluctuations of entropy zero.) Correction of this problem leads to an extra factor of 1/2 in the first (transport) term. Note that this term is quadratic in the gradient. The second (production) term receives no extra factor of (1/2), as it really is a first-order deviation in entropy: the heat production \dot{q} is an externally given function. Fourier's equation for heat conduction is obtained by varying T in

$$\Sigma_{\text{cond}} = \int dV \, (1/2)(kT^2)_* [\nabla(1/T)] + \dot{q}_*/T \quad ,\tag{A.5}$$

holding $(kT^2)_*$ and \dot{q}_* fixed. Afterwards, T_* is self-consistently set equal to T . The heat flux is

$$\mathbf{J}_Q = (l/4\pi r^2)\hat{\mathbf{r}} = -k\nabla T \quad .$$

2. Radiative Diffusion

Radiative transport by photon diffusion is formally similar to heat conduction by matter-matter collisions (Chandrasekhar 1950; Essex 1984a, 1984b; see also: Planck 1913;

Rosen 1954; Kröll 1967; Lallemand and Martinet 1979). The role of k is taken by an expression involving the opacity κ of the matter to photon travel. We should then expect κ to involve an integral over the *photon* phase space. The evaluation of Σ_γ for LTE with a small gradient begins with the *generalized* bilinear form for photons at angular frequency ω passing through and interacting with matter at temperature T :

$$\sigma_{\gamma \text{ diff}} = 2\pi \int_0^\infty d\omega \int_{-1}^{+1} d\xi J_\omega [1/T_\omega - 1/T] \quad , \quad (\text{A.6})$$

where ξ = photon local direction cosine (not to be confused with the reaction rate densities $\dot{\xi}_\alpha$ of section 5), J_ω is the differential radiation luminosity density out of equilibrium:

$$J_\omega = \kappa_\omega \rho [B_\omega - I_\omega] \quad , \quad (\text{A.7})$$

with κ_ω the frequency-specific opacity of matter, B_ω the Planck function (blackbody differential radiation energy flux), and I_ω the true energy flux of photons. In the spherical diffusion approximation, $I_\omega = B_\omega - (\xi/\kappa_\omega \rho)(\partial B_\omega/\partial r)$ with the gradient term small except very near the stellar surface. T_ω is the effective *brightness temperature* for any I_ω and varies with ω :

$$1/T_\omega = (1/\hbar\omega) \ln \left[\frac{2\hbar\omega^3}{8\pi^3 c^2 I_\omega} + 1 \right] \quad . \quad (\text{A.8})$$

In the *Rosseland mean opacity*:

$$1/\kappa \equiv \left(\int_0^\infty d\omega [1/\kappa_\omega] \partial B_\omega / \partial T \right) / \left(\int_0^\infty d\omega \partial B_\omega / \partial T \right) \quad , \quad (\text{A.9})$$

the denominator has the value acT^3 .

The entropy production is a generalization of the bilinear Gibbs formula, with changes in $1/T$ playing the role of the thermodynamic force (which is functionally varied) multiplying some flux, as outlined in Section 3.

3. Source and Boundary Terms

Nuclear burning produces entropy by the production of high-energy photons and the thermalization of fusion products. The radiation/kinetic energy is produced in a fusion reaction by thermalized matter, a tiny, positive contribution to radiation entropy, as the original matter reactants are thermalized. This original photon/kinetic energy is absorbed upon thermalization, a negative contribution to entropy. Both the matter kinetic energy and radiation are then thermalized to the ambient temperature of the core, a large and positive contribution. The first two contributions are negligible compared to the third,

being suppressed by the ratio T/T_0 , where T_0 is the brightness temperature of the original photons, in the range 0.1-5 MeV, well above the typical stellar core temperature. These contribution are small but non-negligible for older stars with higher core temperatures.

For any kind of radiation transport, there is a constant entropy production from the release of radiation into empty space. The local radiation entropy flux \mathbf{H} has magnitude $(4/3)(ac/4)T_{\text{eff}}^3$ on a stellar surface with temperature T_{eff} . Multiplying this expression by the surface area $4\pi R^2$ gives the total boundary entropy production, Σ_{bound} . As in conduction, the transport term arises from second-order fluctuations in the LTE entropy; thus the factor of 1/2 again. The luminosity source term $\varepsilon\rho$ is again a true external first-order deviation from LTE, so the complete entropy production is:

$$\begin{aligned}\Sigma_\gamma &= \Sigma_{\gamma \text{ diff}} + \Sigma_{\text{fusion}} \\ &= \int dV \left[(1/2)[4acT^5/(3\kappa\rho)]_*[\nabla(1/T)]^2 + (\varepsilon\rho)_*/T \right] \quad ,\end{aligned}\tag{A.10}$$

in the radiative diffusion case, where T_* and M are not varied.

A more realistic procedure would be to construct the Σ_γ for the stellar atmosphere. Multiple matter temperatures and a variable radiation brightness temperature must then be introduced. We do not discuss stellar atmospheres in further detail.

A sense of the relative sizes of these entropy production rates may be gotten from the example of the Sun: Conductive heat transport is small; the core and radiative zone transport luminosity by radiative diffusion. We estimate the various entropy production rates, using the boundary contribution $\Sigma_{\text{bound}} \sim 8 \times 10^{29}$ erg/°K sec as a benchmark. The radiation entropy production in the core is much smaller, because of the much higher temperature $T_{\text{eff}} \simeq 5500^\circ\text{K}$ versus $T_c \simeq 15 \times 10^6^\circ\text{K}$: $\Sigma_{\text{fusion}} \simeq (0.0007)\Sigma_{\text{bound}}$. The entropy production rate due to transport of the luminosity from the core to the surface is $\Sigma_{\text{diff}} \simeq (0.0002)\Sigma_{\text{bound}}$. This contribution is also much smaller than the boundary term, but *not* much smaller than the core term. Essentially all of the radiation entropy production in the Sun arises from the release of radiation into empty space at the surface. But, since this term arises from boundary conditions, it is not varied in deriving the transport equation.

Appendix B: Convective Entropy Production

Unlike conductive and radiative transport, convection is a macroscopic process, involving bulk motion of matter. Furthermore, real convection is complicated, with many length scales and the possibility of significant turbulence effects. No satisfactory theory of convection exists, but there are useful models that capture the essentials of the heat transport, enough for consideration in the NESS structure of MS stars (Chandrasekhar

1939; Lamb 1945; Donnelly *et al.* 1966; Glansdorff and Prigogine 1971; Stix 1989; Sieniutycz and Berry 1989, 1991, 1992, 1993; Hansen and Kawaler 1994).

1. Schwarzschild Criterion and Mixing Length Theory

Schwarzschild’s well-known picture begins by idealizing the convective bulk motion of matter cells slightly hotter than their surroundings and heated from below in a gravitational field by a temperature gradient. In the basic *mixing length theory* (MLT), such a cell, with linear size λ_{mix} , moves upwards a distance λ_{mix} before breaking up and merging with the surrounding gas or fluid, releasing its heat in the process. (In real convection, there are many mixing scales, not one.) The cell floats upwards by bouyancy, its internal temperature T' being higher and density ρ' lower than its surroundings. In MLT (Boussinesq approximation), the effects of sound and shock waves are ignored; the pressures internal and external to the cell are assumed equal; and the T and ρ variations between the inside and outside of the cell are small: $(\rho - \rho')/\rho$ and $(T' - T)/T \ll 1$. In stellar applications, in addition, λ_{mix} is assumed much smaller than the stellar radius R (Hansen and Kawaler 1994).

In order for convection to occur at all, the temperature contrast between the bottom of a mixing length, where the cell is heated, and the top must be large enough to overcome gravity by creating a bouyancy force. In the continuum, this leads to Schwarzschild’s criterion for convection to occur:

$$\nabla > \nabla_{\text{ad}} \quad , \quad (\text{B.1})$$

where $\nabla \equiv -d \ln T / d \ln P$, the temperature $T(r)$ and pressure $P(r)$ are the actual profiles of a given star. ∇_{ad} is the same as ∇ , evaluated for the same star, but as if the equation of state were adiabatic, which assumes that the cell exchanges heat at most very slowly with its surroundings before it breaks up. In the convective regime, the temperature gradient in the star outwards is steep enough that relatively hotter cells are bouyant. The simplicity of this condition hides an important complication: the presence or absence of convection itself changes the $T(r)$ and $P(r)$ profiles. In practice, the problem is solved iteratively and self-consistently. For *quasi-adiabatic* convection ∇ is only slightly larger than ∇_{ad} , and the details of this “slow but hot” convection are unimportant. For ∇ well above ∇_{ad} , the specific assumptions of the convection model become important, including turbulence in stellar conditions with very low viscosity (very high Reynolds number) and smallness of λ_{mix} compared to the pressure scale height λ_P . Such convective zones are then “warm but fast”; but, typically being in the outer parts of stars, they are almost irrelevant to the stellar cores.

2. Entropy Production: Heat Loss and Bouyancy

As a convective cell rises, it can lose part of its excess heat by thermal conduction/radiative diffusion (the two can be treated on an equal footing by using the equivalence of Appendix A) and by viscosity. The total entropy production of convection Σ_{conv} is a sum of three terms: the first due to heat loss (positive), the second to viscosity (positive), and the third to bouyancy (negative). The last is negative because bouyancy converts heat to the work of raising the cell in the local gravitational field “g”.

$$\begin{aligned}\sigma_{\text{conv}} &= \sigma_{\text{heat loss}} + \sigma_{\text{visc}} + \sigma_{\text{bouy}} , \\ \sigma_{\text{heat loss}} &= [(\rho c_P)^2 / (2k)] [w \lambda_{\text{mix}} / \lambda_P]^2 (\nabla - \nabla_{\text{ad}})^2 , \\ \sigma_{\text{visc}} &= \eta w^2 / (T \lambda_{\text{mix}}^2) , \\ \sigma_{\text{bouy}} &= -[2(\rho g \gamma_P)(w \lambda_{\text{mix}})^2 / (\eta_T T \lambda_P)] (\nabla - \nabla_{\text{ad}}) ,\end{aligned}\tag{B.2}$$

where w is the upward velocity of the cell. In MLT, w is solved for self-consistently:

$$w = \lambda_{\text{mix}} \sqrt{(\gamma_P g / \lambda_P)(\nabla - \nabla_{\text{ad}})} .$$

The kinematic viscosity is η , the thermal diffusivity $\eta_T \equiv k / \rho c_P$, and c_P is the specific heat capacity of matter at constant pressure. The isobaric exponent $\gamma_P \equiv -(d \ln \rho / d \ln T)_P$. The temperature contrast of the cell (T') with its surroundings (T) is

$$\frac{1}{T} - \frac{1}{T'} = \frac{1}{T} \left(\frac{w}{\lambda_P} \right) \left(\frac{\lambda_{\text{mix}}^2}{\eta_T} \right) [\nabla - \nabla_{\text{ad}}] .\tag{B.3}$$

The convection is *efficient* if the cell’s cooling time $\lambda_{\text{mix}}^2 / \eta_T$ is long compared to its travel time λ_P / w . In stars, the viscosity is still smaller and negligible in MLT, i.e., $\eta \ll \eta_T$.

The inverse temperature difference that acts as the thermodynamic force for heat transfer is just the expression (B.3). The relative temperature contrast, T times equation (B.3), must be small for quasi-linear thermodynamics to be valid. The range of its validity is identical to that of MLT. In the quasi-adiabatic or “slow but hot” regime, $\nabla - \nabla_{\text{ad}}$ is positive but so tiny that the relative temperature contrast is small. In the “warm but fast” regime, $\nabla - \nabla_{\text{ad}}$ is positive and not tiny, and the relative temperature contrast is not small. Quasi-linearity and MLT are then not valid, although MLT is commonly used anyway, for the lack of a better but still simple model. The specific details of convection also matter in this regime, and the simplicity of the quasi-linear situation vanishes.

Convection and convective instability are illuminated by comparing the competing entropy production rates for the *same cell* in two regimes, convective (*heat loss + bouyancy*) and non-convective (*heat loss* alone) (Figure 1) (Donnelly *et al.* 1966; Glansdorff and

Prigogine 1971). Let us start by taking a convective motion with an arbitrary cell velocity w . Without the bouyancy part, the entropy production due to heat conducted out of the cell falls and then rises quadratically, reaching zero at $\nabla = \nabla_{\text{ad}}$ (Figure 1: solid line). (The cell velocity is actually zero in this case and must be replaced by an equivalent expression for pure conduction.) If we put the bouyancy part back in, the entropy production curve is modified by a negative term linear in $\nabla - \nabla_{\text{ad}}$ and becomes distorted (Figure 1: dashed line). The two curves cross at zero, for $\nabla = \nabla_{\text{ad}}$. The branch with lower entropy production for $\nabla < \nabla_{\text{ad}}$ is the *heat loss* alone. But for $\nabla > \nabla_{\text{ad}}$, the *heat loss + bouyancy* branch has the lower entropy production rate and is favored. In this regime, the cell becomes bouyant, and convection begins. The analogy with first-order equilibrium phase transitions is evident, with the entropy production rate taking the place of the free energy.

In the case of convection, there is no variational calculation for the thermal structure, only a comparison of two discrete branches of Σ . The cell velocity w must be computed from hydrodynamic considerations, a topic beyond the scope of this paper.

Appendix C: Neutrino Entropy Production

Because untrapped neutrinos are not in LTE, they do not contribute directly to stellar structure. While the total lepton number (charged leptons plus neutrinos) is conserved, neutrino number is generally not conserved. After their production in the hot stellar core, in quasi-stellar stars, neutrinos suffer negligible weak interactions and change in neutrino number. They therefore transport both neutrino luminosity and number, while electromagnetic radiation transports only photon luminosity. Neutrinos contribute to quasi-static stellar structure only indirectly by cooling of matter.

1. Neutrino Number, Energy, and Entropy Production

In analogy with the specific photon/heat luminosity production rate, a specific neutrino rate, ε_ν , is defined. As for photons, this quantity is a sum over all neutrino-producing reactions,

$$\rho\varepsilon_\nu = \int dV \int dE E \dot{n}_E \quad , \quad (\text{C.1})$$

where \dot{n}_E is the neutrino differential production rate density and we label the neutrino phase space by neutrino energy E , not frequency. The *total* specific luminosity is the function ε . But unlike photons, neutrinos are not thermalized after they are created, leaving the star unimpeded. Their only memory of thermodynamics is the thermalized matter that

produced them. In analogy with the photon entropy production (see Appendix A), the neutrino entropy production may be written:

$$\Sigma_\nu = \int dV \int dE \frac{E \dot{n}_E}{T_E} \quad , \quad (\text{C.2})$$

where \dot{n}_E is the neutrino differential production rate density and we integrate over the single neutrino energy E , instead of frequency. Unlike photons, there is no matter heat bath term for free-streaming neutrinos. Technically, there is a second term in (C.2) giving the tiny contribution to Σ_ν from the neutrino absorption (as in terrestrial detectors). The overall structure of Σ_ν is non-local because neutrinos are generally not in LTE.

Neutrinos have an energy-dependent brightness temperature T_E , but, if unconfined, no chemical potential. This temperature is defined by the Fermi-Dirac analogue of (A.8):

$$\frac{1}{T_E} = \frac{1}{E} \ln \left[\frac{2(E/\hbar c)^3 c}{8\pi^3 I_E} - 1 \right] \quad , \quad (\text{C.3})$$

where $I_E = E \dot{N}_E$ is the energy-specific neutrino differential energy flux; \dot{N}_E is the same for neutrino number. Define additional functions:

$$\begin{aligned} \dot{n}_\nu &= \int dE \dot{n}_E \quad , \quad \dot{N}_\nu = \int dV \dot{n}_\nu \quad , \\ \dot{N}_E &= \int dV \dot{n}_E \quad , \quad N_E = \dot{N}_E / 4\pi r^2 \quad , \end{aligned} \quad (\text{C.4})$$

where the last definition holds only in the case of spherical symmetry. The exact result of (C.4) depends on the specific reaction. (Neutrino entropy production will be further discussed in future publications; see Essex and Kennedy 1996.)

2. Examples

For an infinitely sharp line, unbroadened by thermal fluctuations, the entropy production rate is zero: the neutrinos are produced at exactly one energy. The effect of thermal broadening for neutrino line emission can be treated in the same way as it is for photon emission lines (Stix 1989). Define the thermally broadened shape function $\phi_E(E)$:

$$\phi_E(E) \equiv \frac{H(b, z)}{\sqrt{\pi} \Delta E_D} \quad , \quad (\text{C.5})$$

where $H(b, z)$ is the Voigt function:

$$\begin{aligned} H(b, z) &\equiv (b/\pi) \int_{-\infty}^{\infty} dy \exp(-y^2) / [(z - y)^2 + b^2] \quad , \\ \int_{-\infty}^{\infty} dz H(b, z) &= \sqrt{\pi} \quad . \end{aligned} \quad (\text{C.6})$$

I_E and N_E are proportional to ϕ_E . The reduced variable $z \equiv (E - E_0)/\Delta E_D$, where E_0 is the center of the neutrino line and ΔE_D is the Doppler width:

$$\Delta E_D \equiv E_0 \sqrt{\frac{2T}{m_A c^2} + \frac{\xi_t^2}{c^2}} \quad . \quad (\text{C.7})$$

A is the nucleus emitting the neutrino, and ξ_t is the root-mean-square *microturbulence* speed, set to zero here for simplicity. Define γ as twice the effective collision rate for the nucleus A in the plasma; then:

$$b \equiv \frac{\hbar \gamma}{4\pi \Delta E_D} \quad . \quad (\text{C.8})$$

The collision rate is dominated by collisions of A with the plasma electrons, and, usually, $b \ll 1$. If $b \ll 1$, $H(b, 0) = 1 + \mathcal{O}(b)$.

In the spherical case,

$$\Sigma_\nu = \int dV \int dE [\dot{n}_E]_* \ln \left[\frac{(E/\hbar c)^2 r^2}{\pi^2 \hbar \dot{N}_E} - 1 \right] \quad , \quad (\text{C.9})$$

where the background functions are distinguished again by $*$. This expression is non-local, since we lack neutrino LTE:

$$\dot{N}_E = \int dV' \dot{n}_E(r') \quad .$$

A crude estimate of Σ_ν for a broadened line is:

$$\Sigma_\nu \sim \dot{N}_\nu \ln \left[\frac{(E_0/\hbar c)^3 R_c^2 v_D}{\pi^{3/2} \dot{N}_\nu} \right] \quad , \quad (\text{C.10})$$

where v_D is the Doppler root-mean-square speed: $v_D = c\Delta E_D/E_0$, and R_c is the radius of the neutrino-producing stellar core. In the Sun, for example, the neutrino lines produced by ${}^7\text{Be}$ contribute approximately 5×10^{18} erg/ $^\circ\text{K}$ sec to Σ_ν .

If we let $T \rightarrow 0$ holding all other state variables fixed, the entropy production Σ_ν of a line vanishes as $T^{1/2} \ln T^{1/2}$, as it should: at zero temperature, the reactant matter is not thermalized.

In the continuum case, we distinguish the actual neutrino energy E from the zero-temperature neutrino energy E_0 , which is now continuous. We then define a double differential neutrino production rate density \dot{n}_{EE_0} , per unit dE and per unit dE_0 . Then:

$$\Sigma_\nu = \int dV \int dE_0 \int dE [\dot{n}_{EE_0}]_* \ln \left[\frac{(E/\hbar c)^2 r^2}{\pi^2 \hbar \dot{N}_{EE_0}} - 1 \right] \quad . \quad (\text{C.11})$$

In the case of the Sun's dominant pp -produced neutrinos, we estimate a contribution to Σ_ν of 10^{23} erg/ $^\circ\text{K}$ sec. Because the Sun's photon luminosity so much dominates its neutrino luminosity, the solar neutrino entropy production rate is far smaller than that of solar photons.

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Fig. 1.— Entropy production rate Σ of a potentially convective cell conducting heat only (solid line) versus conducting heat and rising by convection together (dashed line). Convective solution is preferred (lower Σ) if Schwarzschild criterion is satisfied: $\nabla > \nabla_{\text{ad}}$.

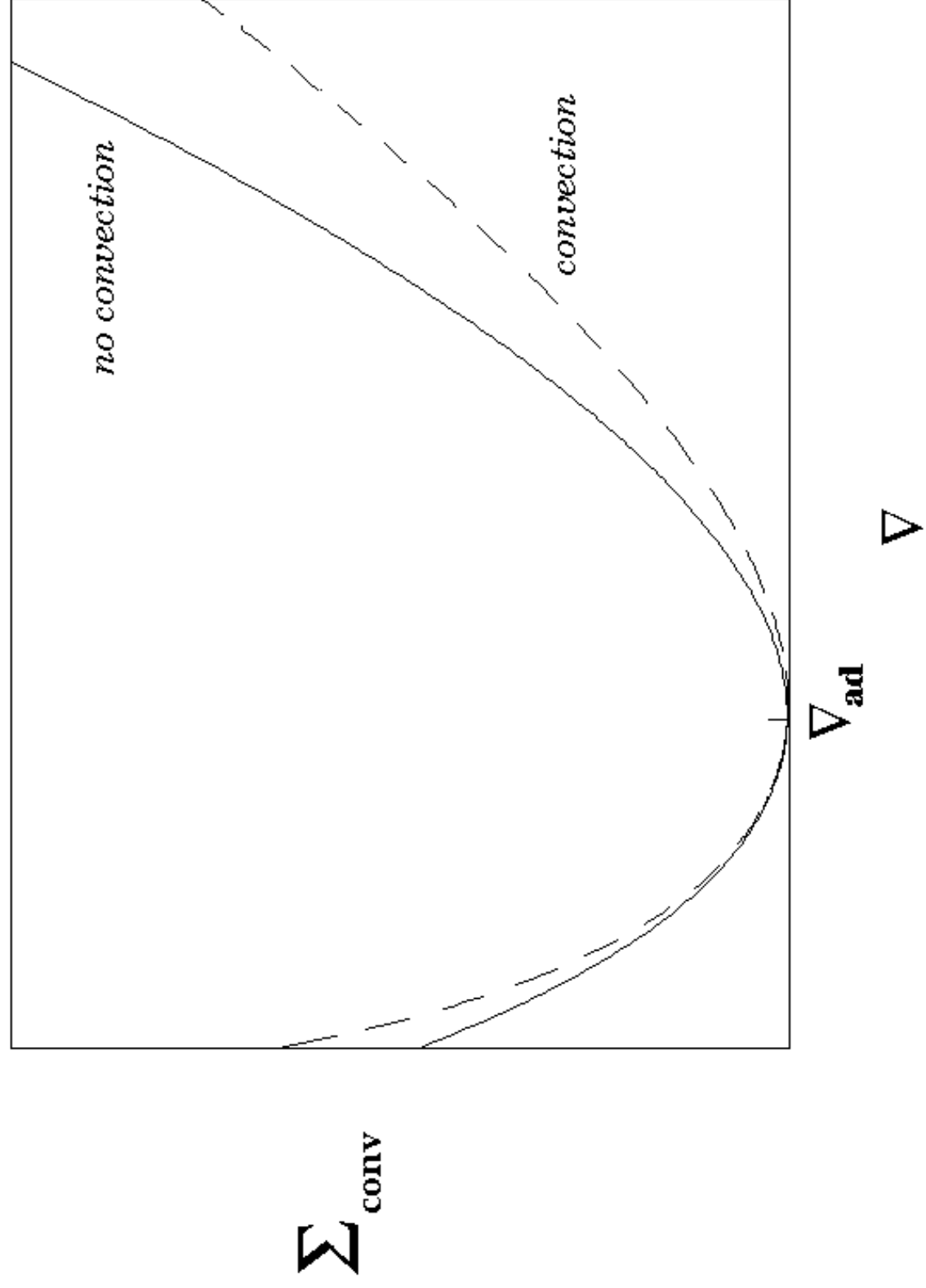


Figure 1